Regularity of inviscid shell models of turbulence

Peter Constantin*
Department of Mathematics, University of Chicago, Chicago, Illinois 60637, USA

Boris Levant†
Department of Computer Science and Applied Mathematics, Weizmann Institute of Science, Rehovot, 76100, Israel

Edriss S. Titi‡
Department of Mathematics and Department of Mechanical and Aerospace Engineering, University of California, Irvine, California 92697, USA and Department of Computer Science and Applied Mathematics, Weizmann Institute of Science, Rehovot, 76100, Israel

(Received 7 July 2006; published 16 January 2007)

In this paper we continue the analytical study of the sabra shell model of energy turbulent cascade. We prove the global existence of weak solutions of the inviscid sabra shell model, and show that these solutions are unique for some short interval of time. In addition, we prove that the solutions conserve energy, provided that the components of the solution satisfy $|u_n| \le C k_n^{-1/3} [\sqrt{n} \log(n+1)]^{-1}$ for some positive absolute constant C, which is the analog of the Onsager's conjecture for the Euler's equations. Moreover, we give a Beal-Kato-Majda type criterion for the blow-up of solutions of the inviscid sabra shell model and show the global regularity of the solutions in the "two-dimensional" parameters regime.

DOI: 10.1103/PhysRevE.75.016304 PACS number(s): 47.10.ad, 47.27.ed, 47.27.Jv

I. INTRODUCTION

Shell models of turbulence have attracted interest as useful phenomenological models that retain certain features of the Navier-Stokes and Euler equations. Their central computational advantage is the parametrization of the fluctuation of a turbulent field in an octave of wave numbers $\lambda^n < |k_n| \le \lambda^{n+1}$ by very few representative variables. This range of wave numbers is called a shell and the variables retained are called shell variables. Like in the Fourier representation of Navier-Stokes equations (NSE), the time evolution of the shell variables is governed by an infinite system of coupled ordinary differential equations with quadratic nonlinearities, with forcing applied to the large scales and viscous dissipation effecting the smaller ones. Because of the very reduced number of interactions in each octave of wave numbers, the shell models are a drastic modification of the original NSE in Fourier space.

The main objective of this work is to investigate the question of existence, uniqueness, and regularity of solutions of the inviscid sabra shell model of turbulence. This model was introduced in [1] and its viscous version was studied analytically in [2]. It is worth noting that the results of this paper apply equally well to the well-known Gledzer-Okhitani-Yamada (GOY) shell model, introduced in [3]. For other shell models see, e.g., [4–6]. A recent review of the subject emphasizing the applications of the shell models to the study of the energy-cascade mechanism in turbulence can be found in [7].

*Email address: const@math.uchicago.edu
†Email address: boris.levant@weizmann.ac.il
‡Email address: etiti@math.uci.edu and edriss.titi@weizmann.ac.il

The sabra shell model of turbulence describes the evolution of complex Fourier-like components of a scalar velocity field denoted by u_n. The associated one-dimensional wave numbers are denoted by k_n, where the discrete index n is referred to as the "shell index." The equations of motion of the viscous sabra shell model of turbulence have the following form:

$$\frac{du_n}{dt} = i(ak_{n+1}u_{n+2}u_{n+1}^* + bk_n u_{n+1}u_{n-1}^* - ck_{n-1}u_{n-1}u_{n-2}) - \nu k_n^2 u_n + f_n, \quad (1)$$

for $n=1,2,3,\ldots$, and the boundary conditions are $u_{-1}=u_0=0$. The wave numbers k_n are taken to be

$$k_n = k_0 \lambda^n, \quad (2)$$

with $\lambda > 1$ being the shell spacing parameter, and $k_0 > 0$. Although the equation does not capture any geometry, we will consider $L = k_0^{-1}$ as a fixed typical length scale of the model. In an analogy to the NSE $\nu > 0$ represents a kinematic viscosity and f_n are the Fourier components of the forcing.

The choice of the nonlinear term in the equation of the sabra model (1) which contains only the local interaction between the shells, can be justified in the context of the Kolmogorov theory of homogeneous turbulence (see [6,8]). The theory states that there is no interchange of energy between the modes of the velocity field with wave numbers separated at least by "an order of magnitude."

The three parameters of the model a, b, and c are real. In order for the sabra shell model to be a system of the hydrodynamic type we require that in the inviscid ($\nu=0$) and unforced ($f_n=0$, $n=1,2,3,\ldots$) case the model will have at least one quadratic invariant. Requiring conservation of the energy

 ©2007 The American Physical Society

$$E = \sum_{n=1}^{\infty} |u_n|^2 \tag{3}$$

leads to the following relation between the parameters of the model, which we will refer to it as the energy conservation condition:

$$a + b + c = 0. \tag{4}$$

Moreover, in the inviscid and unforced case the model possesses (formally) another quadratic invariant

$$W = \sum_{n=1}^{\infty} \left(\frac{a}{c}\right)^n |u_n|^2. \tag{5}$$

The "physically" relevant range of parameters is $|a/c| > 1$ (see [1] for details). For $-1 < \frac{c}{a} < 0$ the quantity W is not sign-definite and therefore it is common to associate it with the helicity—in an analogy to the three-dimensional (3D) turbulence. In that regime we can rewrite the relation (5) in the form

$$W = \sum_{n=1}^{\infty} (-1)^n k_n^{\alpha} |u_n|^2, \tag{6}$$

for

$$\alpha = \log_\lambda \left|\frac{a}{c}\right|. \tag{7}$$

We call the parameters regime corresponding to $0 < \frac{c}{a} < 1$ the two-dimensional (2D) regime. This is because in that case the second conserved quadratic quantity W is non-negative and can be identified with the enstrophy in 2D turbulent flows. We can rewrite the expression (5) in the form

$$W = \sum_{n=1}^{\infty} k_n^{\alpha} |u_n|^2, \tag{8}$$

where α is also defined by Eq. (7).

For the parameters satisfying $\frac{a}{c} = -\lambda$ the sabra shell model becomes "purely three-dimensional," where the quantity (5) scales like the helicity in the 3D Navier-Stokes turbulence. It was found (see [10]) that in that case the energy spectrum in the inertial range of the GOY shell model (this is also true for the sabra model) has the traditional Kolmogorov scaling law $k_n^{-5/3}$. Moreover, while the parameters of the model satisfy $\frac{a}{c} = \lambda^2$, the quantity (6) scales like the enstrophy in the Navier-Stokes 2D turbulence. Therefore these parameters values are usually referred as the "purely two-dimensional" regime. In that case the energy spectrum of the sabra shell model (see [9,10]) obeys the scaling law k_n^{-3}, which is exactly the Kraichnan's law of the 2D developed turbulence.

The famous question of global well-posedness of the 3D Navier-Stokes and Euler equations is a major open problem. In [2] we showed global regularity of weak and strong solutions of Eq. (1) and smooth dependence on the initial data for the case $\nu > 0$. In this work we address the question of existence of regular solutions of the inviscid ($\nu = 0$) sabra shell model (1). First, we prove the global in time existence of weak solutions with finite energy. Similar results for the inviscid GOY shell model were obtained recently in [11]. The existence of weak solutions for inviscid hydrodynamic equations with only energy conservation is not known. The only other analog of the 3D Euler equations known to possess weak solutions of such type is the inviscid surface quasigeostrophic equation (see [12] and [13]). Next, we show that every weak solution $u(t)(u_1(t), u_2(t), \ldots)$ conserves the energy provided that the components of the solution satisfy the decay estimate

$$|u_n| \leq C k_n^{-1/3} [\sqrt{n} \log(n+1)]^{-1},$$

for some positive absolute constant C, namely, provided it is regular enough. A similar result for the solutions of Euler equations is known as the Onsager's conjecture (see [14]) and it was proved in [15] (see also [16,17]). We also give the criteria for the weak solutions to remain unique in certain regularity class.

Next, we show that if the initial data is sufficiently smooth, then the weak solutions are smooth and unique for a short period of time. Similar results were obtained in the context of other discrete models of Euler equations (see, e.g., [18–21]). The well-known Beale-Kato-Majda theorem (see [22,23]) gives a criterion for the blow-up of the initially smooth solutions of the 3D Euler equations. In Sec. V we establish a similar criterion for the inviscid sabra shell model equations and use it to show the global regularity of the solutions of the model in the 2D parameters regime. This picture is consistent with what is known about the global regularity of solutions of the 2D Euler equations (see, e.g., [23–27], and references therein).

Analytic and numerical study of loss of regularity of solutions of the inviscid sabra shell model of turbulence, as well as the dissipation anomaly phenomena in that model, are the subject of ongoing research [28].

II. PRELIMINARIES AND FUNCTIONAL SETTING

We repeat here for the sake of self-consistency the functional settings introduced in Sec. II of [2]. In particular, the Proposition 1 is a slightly more generalized version of the Proposition 1 of [2].

Following the classical treatment of the NSE and Euler equations, and in order to simplify the notation we are going to write the system (1) in the following functional form:

$$\frac{du}{dt} + \nu A u + B(u,u) = f, \tag{9a}$$

$$u(0) = u^{in}, \tag{9b}$$

in a Hilbert space H. The linear operator A as well as the bilinear operator B will be defined below. In our case, the space H will be the sequences space ℓ^2 over the field of complex numbers $\mathbb{C}$. For every $u, v \in H$, the scalar product $(\cdot,\cdot)$ and the corresponding norm $|\cdot|$ are defined as

$$(u,v) = \sum_{n=1}^{\infty} u_n v_n^*, \quad |u| = \left(\sum_{n=1}^{\infty} |u_n|^2\right)^{1/2}.$$

We denote by $\{\phi_j\}_{j=1}^{\infty}$ the standard canonical orthonormal basis of H, i.e., all the entries of ϕ_j are zero except at the place j it is equal to 1.

The linear operator $A:D(A)\to H$ is a positive definite, diagonal operator defined through its action on the elements of the canonical basis of H by

$$A\phi_j = k_j^2 \phi_j,$$

where the eigenvalues k_j^2 satisfy Eq. (2). The space

$$D(A) = \left\{ u \in H: |Au|^2 = \sum_{n=1}^{\infty} k_n^4 |u_n|^2 < \infty \right\}$$

is the domain of A and is a dense subset of H. Moreover, it is a Hilbert space, when equipped with the graph norm

$$\|u\|_{D(A)} = |Au|, \quad \forall\, u \in D(A).$$

Using the fact that A is a positive definite operator, we can define the powers A^s of A for every $s\in\mathbb{R}$

$$\forall u = (u_1,u_2,u_3, \ldots), \quad A^s u = (k_1^{2s}u_1, k_2^{2s}u_2, k_3^{2s}u_3, \ldots).$$

Furthermore, we define the spaces

$$V_s := D(A^{s/2}) = \left\{ u = (u_1,u_2,u_3, \ldots): \sum_{j=1}^{\infty} k_j^{2s}|u_j|^2 < \infty \right\}, \tag{10}$$

which are Hilbert spaces equipped with the scalar product

$$(u,v)_s = (A^{s/2}u, A^{s/2}v), \quad \forall\, u,v \in D(A^{s/2}),$$

and the norm $|u|_s^2=(u,u)_s$, for every $u\in D(A^{s/2})$. Clearly

$$V_s \subseteq V_0 = H \subseteq V_{-s}, \quad \forall\, s>0.$$

Note that the dual space of V_s, for every $s\in\mathbb{R}$, is $V_s'=V_{-s}$. We denote the action of the element in $u\in V_{-s}$ on $v\in V_s$ by

$$\langle u,v\rangle_s = (A^{-s/2}u, A^{s/2}v) = \sum_{n=1}^{\infty} u_n v_n^*.$$

The case of $s=1$ is of a special interest for us. We denote $V=D(A^{1/2})$ a Hilbert space equipped with a scalar product and norm

$$((u,v)) = (A^{1/2}u, A^{1/2}v), \quad \|u\|^2 = ((u,v)),$$

for every $u,v\in V$. The action of the element $u\in V_{-1}=V'$ on $v\in V$ is denoted by

$$\langle u,v\rangle = (A^{-1/2}u, A^{1/2}v) = \sum_{n=1}^{\infty} u_n v_n^*.$$

Before proceeding and defining the bilinear term B, let us introduce the sequence analog of Sobolev functional spaces.

Definition 1. For $1\le p\le\infty$ and $m\in\mathbb{R}$ we define sequence spaces

$$w^{m,p} := \left\{ u=(u_1,u_2,\ldots): \|A^{m/2}u\|_p = \left(\sum_{n=1}^{\infty} (k_n^m|u_n|)^p\right)^{1/p} < \infty \right\},$$

for $1\le p<\infty$, and

$$w^{m,\infty} := \left\{ u=(u_1,u_2,\ldots): \|A^{m/2}u\|_\infty = \sup_{1\le n\le\infty} (k_n^m|u_n|) < \infty \right\}.$$

For $u\in w^{m,p}$ we define its norm

$$\|u\|_{w^{m,p}} = \|A^{m/2}u\|_p,$$

where $\|\cdot\|_p$ is the usual norm in the ℓ^p sequence space. The special case of $p=2$ and $m\ge 0$ corresponds to the sequence analog of the classical Sobolev space, which we denote by

$$h^m = w^{m,2}.$$

Those spaces are Hilbert with respect to the norm defined above and its corresponding inner product.

The above definition immediately implies that $h^d=V^d$, for all d. Moreover, $V_d\subset w^{d,\infty}$ and the inclusion map is continuous because

$$\|u\|_{w^{d,\infty}} = \|A^{d/2}u\|_\infty \le \|A^{d/2}u\|_2 = |u|_d.$$

The bilinear operator $B(u,v)$ will be defined formally in the following way. Let $u,v\in H$ be of the form $u=\sum_{n=1}^{\infty}u_n\phi_n$ and $v=\sum_{n=1}^{\infty}v_n\phi_n$. Then

$$B(u,v) = -i\sum_{n=1}^{\infty} (ak_{n+1}v_{n+2}u_{n+1}^* + bk_n v_{n+1}u_{n-1}^* + ak_{n-1}u_{n-1}v_{n-2} + bk_{n-1}v_{n-1}u_{n-2})\phi_n, \tag{11}$$

where here again $u_0=u_{-1}=v_0=v_{-1}=0$. It is easy to see that our definition of $B(u,v)$ together with the energy conservation condition (4) imply that

$$B(u,u) = -i\sum_{n=1}^{\infty} (ak_{n+1}u_{n+2}u_{n+1}^* + bk_n u_{n+1}u_{n-1}^* - ck_{n-1}u_{n-1}u_{n-2})\phi_n,$$

which is consistent with Eq. (1). In [2] we showed that indeed our definition of $B(u,v)$ makes sense as an element of H, whenever $u\in H$ and $v\in V$ or $u\in V$ and $v\in H$. The next Proposition is a slightly generalized version of Proposition 1 of [2].

Proposition 1.

(i) For all $d,s,\theta\in\mathbb{R}$ and for all $u\in V_{d-\theta+s}$, $v\in V_{d-\theta-s}$, and $w\in w^{1+2\theta,\infty}$

$$|\langle A^d B(u,v),w\rangle_{1+2\theta}| \le C_{d,s,\theta}\|w\|_{w^{1+2\theta,\infty}}|A^{(d-\theta+s)/2}u||A^{(d-\theta-s)/2}v|, \tag{12}$$

where

$$C_{d,s,\theta} = (|a|(\lambda^{1-3d+3\theta+s}+\lambda^{-(1-3d+3\theta+s)}) + |b|(\lambda^{2s}+\lambda^{-(1-3d+3\theta-s)})). \tag{13}$$

(ii) For all $d,s,\theta\in\mathbb{R}$ and for all $w\in V_{d-\theta+s}$, $v\in V_{d-\theta-s}$, and $u\in w^{1+2\theta,\infty}$

$$|\langle A^dB(u,v),w\rangle_{d-\theta+s}|\le c_{d,s,\theta}\|u\|_{w^{1+2\theta,\infty}}|A^{(d-\theta+s)/2}w||A^{(d-\theta-s)/2}v|, \tag{14}$$

where

$$c_{d,s,\theta}=(|a|(\lambda^{-(2d-2s)}+\lambda^{2d-2s})+|b|(\lambda^{1+d-3\theta-s}+\lambda^{1+d+3\theta-s})). \tag{15}$$

(iii) For every $d,s\in\mathbb{R}$, the operator $B:V_{2d-2s}\times V_{1+2s}\to V_{2d}$ and $B:V_{1+2s}\times V_{2d-2s}\to V_{2d}$ is bounded and

$$|A^dB(u,v)|\le\begin{cases}c_{d,s-d,s}\|u\|_{w^{1+2s,\infty}}|A^{d-s}v|,\\ C_{d,s,-1/2}|A^{(d+s)/2+1/4}u||A^{(d-s)/2+1/4}v|,\end{cases} \tag{16}$$

where the constant $C_{d,s,\theta}$ and $c_{d,s,\theta}$ were defined in Eqs. (13) and (15).

(iv) For every $u\in V_d$, $v\in V_{1-2d}$, for all $d\in\mathbb{R}$,

$$\langle B(u,v),u\rangle_d=-\langle B(u,u),v\rangle^*_{1-2d}, \tag{17}$$

and

$$\mathrm{Re}\langle B(v,u),u\rangle_d=0. \tag{18}$$

Proof. To prove the inequality (i), we write

$$\begin{aligned}|\langle A^dB(u,v),w\rangle_{1+2\theta}| &= \left|\sum_{n1}^{\infty}(ak_{n+1}k_n^{2d}v_{n+2}u^*_{n+1}w^*_n+bk_n^{2d+1}v_{n+1}w^*_nu^*_{n-1}+ak_n^{2d}k_{n-1}w^*_nu_{n-1}v_{n-2}+bk_n^{2d}k_{n-1}w^*_nv_{n-1}u_{n-2})\right|\\ &\le\sum_{n=1}^{\infty}|a\lambda^{1-3d+3\theta+s}k_{n+2}^{d-\theta-s}v_{n+2}k_{n+1}^{d-\theta+s}u^*_{n+1}k_n^{1+2\theta}w^*_n|+|b\lambda^{2s}k_{n+1}^{d-\theta-s}v_{n+1}k_n^{1+2\theta}w^*_nk_{n-1}^{d-\theta+s}u^*_{n-1}|\\ &\quad+|a\lambda^{-(1-3d+3\theta+s)}k_n^{1+2\theta}w^*_nk_{n-1}^{d-\theta+s}u_{n-1}k_{n-2}^{d-\theta-s}v_{n-2}|+|b\lambda^{-(1-3d+3\theta-s)}k_n^{1+2\theta}w^*_nk_{n-1}^{d-\theta-s}v_{n-1}k_{n-2}^{d-\theta+s}u_{n-2}|\\ &\le C_{d,s,\theta}\|w\|_{w^{1+2\theta,\infty}}|A^{(d-\theta+s)/2}u||A^{(d-\theta-s)/2}v|.\end{aligned}$$

In the same way we prove the inequality in (ii).

In order to prove the statement (iii) we apply Eq. (14) to obtain the first inequality

$$\begin{aligned}|A^dB(u,v)| &= \sup_{|w|=1}|(A^dB(u,v),w)|\\ &\le\sup_{|w|=1}c_{d,s-d,s}\|u\|_{w^{1+2s,\infty}}|w||A^{d-s}v|\\ &\le c_{d,s-d,s}\|u\|_{w^{1+2s,\infty}}|A^{d-s}v|.\end{aligned}$$

The second inequality is proved similarly.

Finally, the statements (iv) follow directly from the definition of the bilinear operator $B(u,v)$, the energy conservation condition (4), and the inequality (12). □

III. WEAK SOLUTIONS OF THE INVISCID SHELL MODEL

Let us consider the inviscid sabra shell model problem

$$\frac{du}{dt}+B(u,u)=f, \tag{19a}$$

$$u(0)=u^{in}. \tag{19b}$$

One of the main properties of the sabra shell model of turbulence is the locality of the nonlinear interaction. This property allows us to prove the global existence of weak solutions with the finite energy to the inviscid shell model in the following sense. Similar results in the context of the GOY shell model with the stochastic forcing were obtained recently in [11]. In particular, it was shown there that as $\nu\to0$, there exists a subsequence of certain weak solutions of the viscous shell model converging to the weak solution of the inviscid problem, which is also true in our case. In the rest of the section we give the sufficient criteria for the weak solutions to conserve the energy, and investigate the question of the uniqueness of weak solutions.

Definition 2. Let $0<T<\infty$, then $u(t)\in L^\infty([0,T],H)\cap C([0,T],H_w)$ is called a weak solution of the system (19) on the interval $[0,T]$ if for every $0\le t\le T$ it satisfies

$$\langle u(t),v\rangle+\int_0^t\langle B(u(s),u(s)),v\rangle ds=\langle u^{in},v\rangle+\langle f,v\rangle, \tag{20}$$

for every $v\in V$.

Observe that if $u(t)=(u_1(t),u_2(t),u_3(t),\ldots)$, then $u\in C([0,T],H_w)$ is equivalent to $u_n(t)\in C([0,T],\mathbb{C})$, for every $n=1,2,3,\ldots$.

Theorem 1. Let $u^{in},f\in H$, then for every $0<T<\infty$ a weak solution

$$u(t)\in L^\infty([0,T],H)\cap C([0,T],H_w), \tag{21}$$

in the sense of Definition 2 exists. In addition,

$$\frac{du(t)}{dt}\in L^\infty([0,T],V_{-1}). \tag{22}$$

Proof. Let us fix $m>1$. Denote by P_m the orthogonal projection in H onto the first m coordinates and $Q_m = I - P_m$. The Galerkin approximating system of order m for Eq. (19) is an m-dimensional system of ordinary differential equations

$$\frac{du^m}{dt} + P_m B(u^m, u^m) = P_m f, \tag{23a}$$

$$u^m(0) = P_m u^{in}. \tag{23b}$$

First observe that the nonlinear term of Eqs. (23) is quadratic in u^m. Therefore, by the theory of ordinary differential equations, the system (23) has a unique solution on some finite time interval $[0, T_m^*)$. Let us now take the inner product of both sides of Eq. (23a) with u_m and using subsequently inequality (18) of Proposition 1 and the Cauchy-Schwartz inequality we get

$$\frac{1}{2}\frac{d}{dt}|u^m|^2 = (P_m f, u^m) \le |P_m f||u^m| \le |f||u^m|, \tag{24}$$

from which we conclude that

$$|u^m(t)| \le |u^m(0)| + |f|t \le |u^{in}| + |f|t. \tag{25}$$

Therefore u_m is finite in the H norm for all $t < \infty$, hence we can extend the solution of the problem (23) to all $t \in [0, \infty)$.

Let us fix $0 < T < \infty$. Then, from the relation (25) we may conclude that

$$\sup_{0 \le t \le T} |u_n^m(t)| \le C$$

for some constant $C > 0$ depending only on u^{in}, f, and T. Moreover, writing Eq. (23a) in the componentwise form

$$u_n^m(t) = u_n^m(0) + \int_0^t i(ak_{n+1}u_{n+2}^m (u_{n+1}^m)^* + bk_n u_{n+1}^m (u_{n-1}^m)^* + ak_{n-1}u_{n-1}^m u_{n-2}^m + bk_{n-1}u_{n-1}^m u_{n-2}^m) ds + f_n. \tag{26}$$

For $0 \le t \le T$, we get that for every n there exists a constant C_n, independent of m, such that

$$\|u_n^m\|_{C^1([0,T],\mathbb{C})} \le C_n.$$

Applying the Arzela-Ascoli theorem we conclude that for every n there exists a subsequence $(m_k^n)_{k\ge 1}$ such that $u_n^{m_n^k}$ converges uniformly to some u_n, as $k \to \infty$. Moreover, by a diagonalizing procedure we can choose a sequence $(m_k)_{k\ge 1}$, independent of n such that $u_n^{m_k}$ converges uniformly to $u_n \in C([0,T],\mathbb{C})$ and we denote

$$u(t) = (u_1(t), u_2(t), u_3(t), \dots).$$

Using the uniform convergence it is easy to show, passing to the limit in the expression (26), that $u(t)$ satisfies the weak form of the sabra shell model equation in the form

$$\langle u(t), v^n\rangle + \int_0^t \langle B(u(s), u(s)), v^n\rangle ds = \langle u^{in}, v^n\rangle + \langle f, v^n\rangle, \tag{27}$$

for every $v^n \in H$ with the finite number of components different from zero.

Moreover, we need to show that $u(t) \in L^\infty([0,T],H)$. The sequence $\{u^m\}_{m\ge 1}$ is uniformly bounded in H [see Eq. (25)], and hence

$$u^m \text{ is bounded in every } L^p([0,T],H), \quad \text{for } 1 \le p \le \infty. \tag{28}$$

Therefore we conclude that there exists a subsequence $(m_k)_{k\ge 1}$ such that u^{m_k} converges to $w(t)$ in the weak-* topology of $L^\infty([0,T],H)$, and by definition it is not hard to see that the limiting function is indeed $w(t) \equiv u(t)$. In addition, by the inequality (iii) of Proposition 1, we get

$$|B(u,u)|_{-1} = |A^{-1/2}B(u,u)| \le C|u|^2,$$

for $C = C_{-1/2,0,-1/2}$ [see Eq. (13)], concluding that $B(u,u) \in L^\infty([0,T], V_{-1})$.

Finally, let $v \in V$, and $v^n = P_n v$, with the finite number of components being different from zero, converging strongly to v. Then letting $n \to \infty$ in the relation (27) we conclude that $u(t)$ satisfies Eq. (19) in the weak sense of Definition 2. □

The next Theorem gives a partial answer to the question: under which conditions do the weak solutions conserve the energy?

Theorem 2. Let $u(t)$ be a weak solution, whose existence is proved in Theorem 1, satisfying

$$u(t) \in L^\infty([0,T], V_{1/3}), \tag{29}$$

for some $T > 0$. Then for every $t \in [0,T]$

$$|u(t)|^2 = |u^{in}|^2 + \int_0^t (f, u(s)) ds. \tag{30}$$

Proof. If a weak solution satisfies Eq. (29), then according to the inequality (iv) of Proposition 1, Eq. (19) can be considered as the equation in the space $V_{-1/3}$. Applying the operator $A^{-1/3}$ to both sides of Eq. (19) and taking an inner product with $A^{1/3}u(t)$ in the space H we get, using the identity (iv) of Proposition 1,

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 = (f, u(t)),$$

from which the statement follows. □

As we already mentioned in the Introduction, our result for the sabra shell model of turbulence is reminiscent of Onsager's conjecture for the Euler equations (see [14–17]). However, the criterion given by Theorem 2 is not sharp. It is easy to give an example of a solution of the inviscid sabra shell model of turbulence, which stays merely in H, but still conserves the energy. To see this, consider the forcing $f = (f_1, f_2, \dots)$, where

$$f_n = \begin{cases} \frac{1}{n}, & n = 1, 3, 6, \dots, \\ 0, & o/w. \end{cases}$$

Then solution $u(t) = (u_1(t), u_2(t), \dots)$, where $u_n(t) = \frac{t}{n}$, for $n = 1, 3, 6, \dots$, is a weak solution of the sabra shell model, corresponding to the zero initial condition. The function $u(t)$ is only in H for every $t < \infty$, however, it is easy to see that it conserves the energy. Clearly, this example is pathological in a sense that all nonlinear interactions are absent due to the wide gaps between the excited modes, however, it shows that the result of Theorem 2 is not sharp. Moreover, it is not known when the weak solutions of the inviscid sabra shell model dissipate energy. These questions will be studied in the forthcoming work [28].

The final result of this section gives criterions for the uniqueness of weak solutions.

Theorem 3.

(i) Let $u(t),v(t)$ be two weak solutions, whose existence is proved in Theorem 1, satisfying

$$u(t),v(t) \in L^1([0,T],w^{1,\infty}), \tag{31}$$

for some $T>0$, and $u(0)=v(0)$. Then $u(t)=v(t)$, for all $t\in[0,T]$.

(ii) If $u(t),v(t)$ are two weak solutions, satisfying

$$u(t) \in L^1([0,T],w^{1,\infty}) \cap L^\infty([0,T],V_{1/3}), \tag{32}$$

and

$$v(t) \in L^\infty([0,T],V_{1/3}), \tag{33}$$

for some $T>0$, with $u(0)=v(0)$. Then $u(t)=v(t)$, for all $t\in[0,T]$.

Proof. First, let $u,v\in L^1([0,T],w^{1,\infty})$ be two weak solutions of the inviscid sabra shell model with the same initial conditions. Denote $w=u-v$ satisfying

$$\frac{dw}{dt}+B(u,w)+B(w,v)=0,$$

with $w(0)=0$. Using the fact that u,v, and w satisfy Eqs. (22) and (21), we are allowed to apply the operator $A^{-1/2}$ to both sides of the last equation and then take the inner product of both sides with w in H to conclude

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|A^{-1/4}w|^2 &\le |(A^{-1/2}B(u,w),w)|+|(A^{-1/2}B(w,v),w)|\\ &= |(A^{-1/2}B(u,w),w)|+|(B(w,A^{-1/2}w),v)|\\ &\le C_1(\|v\|_{w^{1,\infty}}+\|u\|_{w^{1,\infty}})|A^{-1/4}w|^2,\end{aligned} \tag{34}$$

where we subsequently used parts (iv), (i), and (ii) of Proposition 1 and $C_1=C_{-1/2,-1/2,0}+c_{-1/2,0,0}$. Applying Gronwall's inequality to Eq. (34) we get

$$|A^{-1/4}w(t)|^2 \le |A^{-1/4}w(0)|^2 e^{C_1\int_0^t[\|v(s)\|_{w^{1,\infty}}+\|u(s)\|_{w^{1,\infty}}]ds},$$

for $t\in[0,T]$, concluding the proof of part (i).

To prove part (ii) of the theorem, let $u(t)$ be the solution of the inviscid sabra shell model satisfying Eq. (32). Let $v(t)$, satisfying Eq. (33), be another weak solution with the same initial data $v(0)=u(0)$. Note that, in particular, both $u(t)$ and $v(t)$ conserve the energy, according to Theorem 2. Denote $w=u-v$ satisfying

$$\frac{dw}{dt}+B(u,w)+B(w,u)+B(w,w)=0, \tag{35}$$

with $w(0)=0$. Just as in the proof of Theorem 2, we can consider Eq. (35) as an equation in the space $V_{-1/3}$. Therefore, applying the operator $A^{-1/3}$ to both sides of Eq. (19), and taking an inner product with $A^{1/3}w(t)$ in the space H we get, using the parts (i) and (iv) of Proposition 1,

$$\frac{1}{2}\frac{d}{dt}|w|^2 \le |\langle B(w,u),w\rangle_{1/3}| \le C_2\|u\|_{w^{1,\infty}}|w|^2,$$

for $C_2=C_{0,0,0}$. It follows that

$$|w(t)|^2 \le |w(0)|^2 e^{C_2\int_0^t\|u(s)\|_{w^{1,\infty}}ds},$$

for $t\in[0,T]$, finishing the proof of the theorem. □

IV. THE SHORT-TIME EXISTENCE AND UNIQUENESS OF STRONG SOLUTIONS

The uniqueness of the weak solutions, whose existence was proved in Theorem 1, is not known. In this section we prove that the weak solutions of the inviscid case sabra shell model are unique as long as they stay smooth enough, at least on the short-time interval $[0,T_*)$, where the time T_* depends on the parameters of the problem (a,b,c,k_0, and λ), as well as the initial data u^{in} and f. Let us consider the inviscid sabra shell model problem (19) as an ordinary differential equation (ODE) in the Hilbert space V_d, for $d\ge 1$. The main theorem of this section shows the short time existence and uniqueness of solutions of Eq. (19).

Theorem 4. Let $u^{in}\in V_d$ and $f\in V_d$ for some $d\ge 1$.

(i) There exists a time $T>0$, such that the inviscid problem (19) has a unique solution $u(t)$ satisfying

$$u(t) \in C^1((-T,T),V_d).$$

(ii) Moreover, if $f\in V_{2d-1}$, then

$$\frac{du}{dt} \in C((-T,T),V_{2d-1}).$$

(iii) The unique solution to the inviscid sabra shell model (19) either exists globally in time, or there exists a maximal positive time of existence $T_*>0$ such that

$$u(t) \in C^1([0,T_*),V_d),$$

and

$$\limsup_{t\to T_*^-}|u(t)|_d = \infty .$$

A similar statement can be formulated for the maximal negative time of existence.

In our notation $C^1((-T,T),V_d)$ denotes continuously differentiable functions on the interval $(-T,T)$ with values in V_d. Our proof is based on the classical Picard theorem for ODEs in Banach spaces (see, for example, [23,29,30]).

Proof. (Of Theorem 4)

Let us write the system (19) in the form

$$\frac{du}{dt} = F(u), \quad u(0) = u^{in}, \tag{36}$$

where $F(u)=f-B(u,u)$. Fix $d \geq 1$, then according to part (iii) of Proposition 1 the operator $B(u,u)$ maps V_d into $V_{2d-1} \subseteq V_d$. Hence the mapping $F(u)$ maps V_d into itself. Moreover, for every $u,v \in V_d$ we have the following estimates:

$$|F(u) - F(v)|_d = |B(u,u) - B(v,v)|_d \leq |B(u-v,u)|_d + |B(v,u-v)|_d \leq C_3(\|u\| + \|v\|)|u-v|_d,$$

where the last inequality follows from relation (16), and $C_3 = c_{d/2,-1/2,(d-1)/2} + c_{d/2,-d/2,0} > 0$ [see Eq. (15)]. Therefore, we conclude that the mapping $F(u)$ is locally Lipschitz continuous, and we are able to apply the Picard theorem, completing the proof.

Part (iii) follows by the straightforward application of classical theory of ODEs. To prove part (ii) we apply inequality (16) to both sides of Eq. (19). □

Using part (iii) of Theorem 4 we will be able in Sec. V to derive a criterion for the blow-up of the solutions of the inviscid sabra shell model and to prove the global (in time) existence of the unique, regular solutions in the particular case of the 2D regime of the inviscid sabra shell model.

V. A BEALE-KATO-MAJDA TYPE RESULT

The Beale-Kato-Majda theorem (see [22,23]) states, citing the original paper, "if a solution of the Euler or Navier-Stokes equations is initially smooth and loses its regularity at some later time, then the maximum vorticity necessarily grows without bound as the critical time approaches." More precisely, if the initially smooth solution of the Euler equations cannot be continued beyond the time T^*, and T^* is the first such time, then

$$\lim_{t\to T^{*-}} \int_0^t \|\omega(\cdot,s)\|_{L^\infty} ds = \infty,$$

where $\omega = \operatorname{curl} v$ is the vorticity and v is the velocity field of the Euler equations.

Our goal in this section is to derive a similar criterion for the loss of regularity of the solutions of the inviscid sabra shell model. Note that in our case, the analog of the L^∞ norm of the vorticity would be the ℓ^∞ norm of the velocity "derivative," namely

$$\|u\|_{w^{1,\infty}} = \sup_{1\leq n\leq\infty} k_n |u_n|.$$

Clearly, if this quantity becomes infinite at some finite moment of time, then all higher norms, namely $|u|_d$, for $d \geq 1$, become unbounded at the same time. However, in the spirit of the Beale-Kato-Majda result for the Euler equations, we show that the opposite is also true. In other words, we show that if a regular solution $u(t)$ of the inviscid shell model problem loses its regularity for the first time at the time T, then

$$\int_0^t \|u(s)\|_{w^{1,\infty}} ds \to \infty, \quad \text{as } t \to T^-.$$

For simplicity we would like to focus on the inviscid sabra shell model problem (19) without forcing

$$\frac{du}{dt} + B(u,u) = 0, \tag{37a}$$

$$u(0) = u^{in}(x). \tag{37b}$$

Theorem 5. Let $u^{in} \in V_d$, for some $d \geq 1$. Let $u(t) \in C^1([0,T_*),V_d)$ be the solution of the inviscid shell model equation (37), where T_* is its maximal positive time of existence. Then, either $T_* = \infty$ or

$$\lim_{t\to T_*^-} \int_0^t \|u(\tau)\|_{w^{1,\infty}} d\tau = \infty,$$

and hence

$$\limsup_{t\to T_*^-} \|u(t)\|_{w^{1,\infty}} = \infty.$$

Proof. Let us fix $d \geq 1$ and consider $u(t)$—the unique solution to the sabra shell model equation (37). According to part (ii) of Theorem 4, both sides of Eq. (37a) lie in the space V_{2d-1}. Therefore we are allowed to apply the operator $A^{d/2}$ to both sides of the equation and take the inner product in H with $A^{d/2}u \in H$. After taking the real part we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u|_d^2 &= \operatorname{Re}(A^{d/2}B(u,u),A^{d/2}u) \\ &= -\operatorname{Im}\sum_{n=1}^{\infty} (ak_{n+1}k_n^{2d}u_{n+2}u_{n+1}^*u_n^* + bk_n^{2d+1}u_{n+1}u_n^*u_{n-1}^* \\ &\quad + ak_n^{2d}k_{n-1}u_n^*u_{n-1}u_{n-2} + bk_n^{2d}k_{n-1}u_n^*u_{n-1}u_{n-2}) \\ &= E_d \operatorname{Im}\sum_{n=2}^{\infty} k_n^{2d+1}u_{n+1}u_n^*u_{n-1}^*,\end{aligned} \tag{38}$$

which we denote by

$$E_d = a(\lambda^{2d} - \lambda^{-2d}) + b(\lambda^{2d} - 1). \tag{39}$$

Applying Cauchy-Schwarz inequality to Eq. (38) we get

$$\frac{d}{dt}|u|_d^2 \leq 2|E_d| \|u\|_{w^{1,\infty}} |u|_d^2,$$

which is valid for $t \in [0,T]$, and for every $T < T_*$. Using Gronwall's inequality we conclude

$$|u(t)|_d \leq |u(0)|_d e^{|E_d| \int_0^T \|u(\tau)\|_{w^{1,\infty}} d\tau}. \tag{40}$$

The Theorem follows after letting $T \to T_*^-$ in the last inequality. □

VI. THE "TWO-DIMENSIONAL" REGIME

Recall that for the range of parameters satisfying $0 < c/a < 1$, corresponding to the "two-dimensional" regime, the sabra shell model possesses two different positive quadratic invariants, one is associated with the energy and the second is associated with the enstrophy in the analogy with the 2D Euler equation (see [9])

$$W = |A^{d_0/2}u| = |u|_{d_0}, \tag{41}$$

where

$$d_0 = \frac{1}{2}\log_\lambda\left(\frac{a}{c}\right). \tag{42}$$

In this case E_{d_0}, defined in Eq. (39), equals 0. Moreover, if $d_0 \geq 1$, the following inequality holds:

$$\|u\|_{w^{1,\infty}} \leq |u|_{d_0},$$

for every $u \in V_{d_0}$. From the relation (42) we conclude that the condition $d_0 \geq 1$ corresponds to the case when parameters of the inviscid sabra shell model satisfy

$$0 < \frac{c}{a} \leq \lambda^{-2}. \tag{43}$$

It is well-known that the Euler equations of the ideal incompressible fluid in 2D possess a global in time, unique, regular solution (see, for example, [23–27,31]). The same statement is true for the inviscid shell model of turbulence, namely.

Corollary 1. (Global Existence) Let d_0 be defined by the relation (42).

(i) Let the parameters a, c, and λ of the inviscid sabra shell model (37) satisfy

$$\frac{c}{a} > \lambda^{-2}.$$

Then for $u^{in} \in V_{d_0}$, there exists a weak solution $u(t)$ to the inviscid problem (37) satisfying

$$u(t) \in L^\infty((-\infty,\infty),V_{d_0}).$$

(ii) The weak solution $u(t)$ conserves the enstrophy (41), for all $t \in [0,T]$, provided

$$u(t) \in L^\infty([0,T],V_{(1-2d_0)/3}).$$

(iii) If the parameters a, c, and λ of the inviscid sabra shell model (37) satisfy the relation (43), then for $u^{in} \in V_s$, $s \geq d_0$, there exists a unique global solution $u(t)$ to the inviscid problem (37) satisfying

$$u(t) \in C^1((-\infty,\infty),V_s).$$

The proof of part (i) of Corollary 1 is essentially the same as the proof of Theorem 1. The proof of part (i) is similar to that of Theorem 2, and part (iii) follows from the criterion, proved in Theorem 5.

The comprehensive numerical study of the shell model of turbulence in the 2D parameters regime was performed in [9] for the sabra model, and previously in [10] for the GOY model. In particular, it showed that parameters setting defined by relation (43) corresponds to the enstrophy and energy equipartition across the inertial range. Therefore our rigorous result on existence of the solutions of the inviscid shell model (37) globally in time supports these findings.

It was also found numerically that for the parameters satisfying $\frac{c}{a} > \lambda^{-2}$, the shell models exhibit the direct enstrophy cascade in the inertial range and the energy distribution becomes close to the Kraichnan's dimensional prediction

$$\langle |u_n|^2\rangle \sim k_n^{-2/3[1+\log_\lambda(a/c)]},$$

with small corrections, for $n_f \ll n \ll n_d$, where n_f is the largest wave number of the forcing and n_d is the Kraichnan's dissipation wave number (see [10]). Note that for $\frac{c}{a} = \lambda^{-2}$ this estimate exactly coincides with the well-known prediction k_n^{-3} for the energy spectrum of the 2D developed turbulence (see [32]). Using these reasonings we conclude that for

$$\frac{c}{a} > \lambda^{-2}, \tag{44}$$

the inertial range of the sabra shell model with nonzero viscosity will scale like $\langle |u_n|^2\rangle \sim k_n^{-2+\delta}$, for some positive δ. Therefore it is natural to expect that if the viscosity tends to zero, or equivalently, the dissipation scale n_d tends to infinity, the solutions of the inviscid sabra shell model, for parameters satisfying relation (44), will blow-up in finite time, for some initial conditions, according to the criterion proved in Theorem 5.

VII. CUBIC INVARIANT AND HAMILTONIAN STRUCTURE

In practical numerical simulations of the sabra shell model one is limited to consider a truncated model of N equations, setting $u_n = 0$, for $n = N+1, N+2, \ldots$. It was shown in [33] that such a finite system in the inviscid and unforced case possesses a Hamiltonian structure for a specific value of the parameters. In this section we will state, based on our results on the existence of the solutions of the inviscid sabra shell model, that the infinite system of equations also has a Hamiltonian structure.

By rescaling the time and taking into account the energy conservation assumption (4), we will assume that

$$a = 1,\ b = -\epsilon,\ c = \epsilon - 1.$$

Let us fix $\epsilon = (\sqrt{5}-1)/2$—the golden mean satisfying $\epsilon^2 = 1-\epsilon$. In that case we can rewrite Eqs. (1) in the equivalent form

$$\frac{du_n}{dt} = ik_{n+1}\left(u_{n+2}u_{n+1}^* - \frac{\epsilon}{\lambda}u_{n+1}u_{n-1}^* + \frac{\epsilon^2}{\lambda^2}u_{n-1}u_{n-2}\right) - \nu k_n^2 u_n + f_n, \tag{45}$$

for $n = 1,2,3,\ldots$. In that case the inviscid sabra shell model without forcing has, formally, a cubic invariant of the form

$$I = \sum_{n=1}^{\infty} \epsilon k_0 \left(-\frac{\lambda}{\epsilon}\right)^n (u_{n+1}^* u_n u_{n-1} + \text{c.c.}), \tag{46}$$

where c.c. stands for complex conjugate.

Following the method of [33], we perform the following change of variables:

$$a_n = \frac{u_n}{\epsilon^{n/2}}, \quad \text{for even } n,$$

and

$$a_n = -\frac{u_n^*}{\epsilon^{n/2}}, \quad \text{for odd } n.$$

The modified equations then take the form of

$$\frac{da_n}{dt} = -ik_0\epsilon(\lambda\sqrt{\epsilon})^n\left(\lambda\sqrt{\epsilon}a_{n+2}a_{n+1} + a_{n+1}^*a_{n-1} + \frac{1}{\lambda\sqrt{\epsilon}}a_{n-1}^*a_{n-2}\right), \tag{47}$$

for $n=1,2,3,\ldots$. Finally, the Hamiltonian takes the form

$$\mathcal{H} = \sum_{n=1}^{\infty}\mathcal{H}_n, \tag{48}$$

where

$$\mathcal{H}_n = \epsilon k_0(\lambda\sqrt{\epsilon})^n(a_{n+1}a_na_{n-1}^* + \text{c.c.}). \tag{49}$$

In order to see that $\mathcal{H}$ is indeed a Hamiltonian we note that

$$\frac{d\mathcal{H}}{dt} = 0,$$

and the equations of motion (47) satisfy

$$\frac{da_n}{dt} = -i\frac{\partial\mathcal{H}}{\partial a_n^*}, \quad \frac{da_n^*}{dt} = i\frac{\partial\mathcal{H}}{\partial a_n}.$$

Both the cubic invariant I and the Hamiltonian $\mathcal{H}$ are defined by infinite sums. Therefore it is natural to ask when those definitions make sense, namely when the sums converge.

Lemma 1. For $\lambda^2 \geq \epsilon^{-1}$ the Hamiltonian (48) is well-defined for all $u \in V$.

Proof. The Lemma follows by a simple application of Hölder inequality. □

Finally, we can conclude the following.

Corollary 2. Let the parameters of the inviscid sabra shell model (37) satisfy

$$a=1, \quad b=-\frac{\sqrt{5}-1}{2}, \quad c=\frac{\sqrt{5}-3}{2}, \quad \lambda^2 \geq \frac{2}{\sqrt{5}-1}.$$

Then the inviscid sabra shell model (37) with initial data in V_d, for $d \geq 1$, is a Hamiltonian system with the Hamiltonian defined by relation (48), as long as a solution of the model exists.

VIII. CONCLUSIONS

In this work we continued the analytic study of the shell models of turbulence, initiated in [2]. We established the global existence of weak solutions and showed that strong solutions remain regular and unique for some short period of time. Moreover, we showed that the solutions for the "two-dimensional" range of parameters remain regular and unique globally in time. In addition, we established a Beale-Kato-Majda type criterion for the blow-up of the initially smooth solutions.

We showed that for some parameter regime the sabra shell model is an infinite dimensional Hamiltonian system. In contrast to the Euler equations, which possess a quadratic Hamiltonian function (see, for example, [34,35]), the Hamiltonian of the inviscid sabra shell model is cubic.

We showed that the weak solution $u(t)=(u_1(t),u_2(t),\ldots)$ conserve the energy provided that the components of the solution satisfy

$$|u_n| \leq Ck_n^{-1/3}[\sqrt{n}\log(n+1)]^{-1}$$

for some positive absolute constant C. A similar result for the Euler equations is known as the Onsager's conjecture (see [14]) and it was proved in [15] (see also [16,17]). The question of whether less regular solutions dissipate energy remains open.

The question of the possible loss of regularity and uniqueness for the initially smooth solutions of the inviscid sabra shell model outside of the "two-dimensional" range of parameters still remains open.

The dimensional argument for the viscous ($\nu>0$) GOY shell model, which is also applicable to the sabra model, indicates that in the "three dimensional" parameters regime $-1<\frac{c}{a}<0$ the velocity field scales like

$$\langle|u_n|\rangle \sim k_n^{-1/3(1+\log_\lambda|a/c|)},$$

at the inertial range (see [10]). Therefore at least for the parameters regime satisfying

$$\left|\frac{c}{a}\right| > \lambda^{-2}, \tag{50}$$

we might expect the blow-up of the inviscid sabra (as well as GOY) shell model of turbulence.

We would like to mention that the techniques used to prove the blow-up for other discrete models of Euler equations (see, e.g., [18–21,36]) could not be applied directly in the case of the sabra shell model of turbulence. The study of possible loss of regularity of the inviscid sabra shell model in different parameters regime is the subject of ongoing work ([28]).

ACKNOWLEDGMENTS

The authors would like to thank I. Procaccia, V. Lvov, and A. Pomyalov for the very stimulating and inspiring discussions. E.S.T. is thankful to the kind hospitality of the École Normale Supérieure - Paris where this work was completed. The work of P.C. was partially supported by the NSF Grant No. DMS-0504213. The work of E.S.T. was supported in part by the NSF Grant No. DMS–0504619, the Israeli Council of Higher Education, and by the BSF Grant No. 2004271.

[1] V. S. L'vov, E. Podivilov, A. Pomyalov, I. Procaccia, and D. Vandembroucq, Phys. Rev. E **58**, 1811 (1998).
[2] P. Constantin, B. Levant, and E. S. Titi, Physica D **216**, 120 (2006).
[3] K. Okhitani and M. Yamada, Prog. Theor. Phys. **89**, 329 (1989).
[4] T. Bohr, M. H. Jensen, G. Paladin, and A. Vulpiani, *Dynamical Systems Approach to Turbulence* (Cambridge University Press, Cambridge, England, 1998).
[5] E. B. Gledzer, Sov. Phys. Dokl. **18**, 216 (1973).
[6] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, England, 1995).
[7] L. Biferale, Annu. Rev. Fluid Mech. **35**, 441 (2003).
[8] G. Gallavotti, *Foundations of Fluid Dynamics* (Springer-Verlag, Berlin, 2002).
[9] T. Gilbert, V. S. L'vov, A. Pomyalov, and I. Procaccia, Phys. Rev. Lett. **89**, 074501 (2002).
[10] P. D. Ditlevsen and I. A. Mogensen, Phys. Rev. E **53**, 4785 (1996).
[11] D. Barbato, M. Barsanti, H. Bessaih, and F. Flandoli (unpublished).
[12] P. Constantin, D. Cordoba, and J. Wu, Indiana Univ. Math. J. **50**, 97 (2001).
[13] S. Resnick, Ph.D. thesis, University of Chicago, Chicago, 1995 (unpublished).
[14] L. Onsager, Nuovo Cimento, Suppl. **6**, 279 (1949).
[15] P. Constantin, W. E, and E. S. Titi, Commun. Math. Phys. **165**, 207 (1994).
[16] J. Duchon and R. Robert, Nonlinearity **13**, 249 (2000).
[17] G. Eyink, Physica D **78**, 222 (1994).
[18] S. Friedlander and N. Pavlović, Commun. Pure Appl. Math. **57**, 705 (2004).
[19] N. Katz and N. Pavlović, Trans. Am. Math. Soc. **357**, 695 (2005).
[20] A. Kiselev and A. Zlatoš, IMRN **38**, 2315 (2005).
[21] F. Waleffe, Proc. Am. Math. Soc. **134**, 2913 (2006).
[22] J. T. Beale, T. Kato, and A. Majda, Commun. Math. Phys. **94**, 61 (1984).
[23] A. J. Majda and A. L. Bertozzi, *Vorticity and Incompressible Flow* (Cambridge University Press, Cambridge, England, 2002).
[24] C. Bardos, Bull. Soc. Math. France, Suppl., Mém. **31-32**, 39 (1972).
[25] C. Bardos, J. Math. Anal. Appl. **40**, 769 (1972).
[26] C. Marchioro and M. Pulvirenti, *Mathematical Theory of Incompressible Nonviscous Fluids* (Springer-Verlag, New-York, 1994).
[27] V. I. Yudovich, Zh. Vychisl. Mat. Mat. Fiz. **3**, 1032 (1963).
[28] P. Constantin, B. Levant, and E. S. Titi (unpublished).
[29] P. Hartman, *Ordinary Differential Equations* (Birkhäuser, Boston, 1982).
[30] M. Schechter, *An Introduction to Nonlinear Analysis* (Cambridge University Press, Cambridge, England, 2004).
[31] T. Kato (unpublished).
[32] R. H. Kraichnan, Phys. Fluids **10**, 1417 (1967).
[33] V. S. L'vov, E. Podivilov, and I. Procaccia, Europhys. Lett. **46**, 609 (1999).
[34] V. I. Arnold and B. Khesin, *Topological Methods in Hydrodynamics* (Springer-Verlag, New York, 1998).
[35] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry* (Springer-Verlag, New York, 1999).
[36] A. Cheskidov, AMS Trans. (to be published).